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THREE-DIMENSIONAL FINITE ELEMENT MODELING OF INDUCTIVE AND CAPACITIVE EFFECTS IN MICRO-COILS

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***Abstract* – Dual finite element formulations are developed for modeling both inductive and capacitive effects in massive inductors. Mixed finite elements are used to satisfy each chosen conformity level for the unknown fields and to naturally define the global quantities involved in the inductive and capacitive circuit relations, to be used in circuit coupling. The interest of satisfying conformity properties for the considered magnetic and electric coupled problems is shown and the related mathematical and discrete tools for any wished conformity level are developed.**

Introduction

Beyond a certain level of frequency and for certain configurations of inductors, the capacitive effects cannot be neglected versus the inductive ones. The circuit relation describing such inductors is not only defined by a resistance and an inductance in series, but has to be extended with a capacitance in parallel as well [1]-[4].
Dual three-dimensional finite element (FE) formulations are developed to couple both inductive and capacitive effects in inductors, in particular micro-coils. The coupling is done through the definition of circuit relations involving a unique voltage and complementary inductive and capacitive currents. The inductive circuit relation is first classically obtained by a magnetodynamic model [5], [6]. Then, the capacitive relation is obtained through an electric model, using sources evaluated from the first model.
A particular attention is paid to the FE approximations of the unknowns, keeping the same conformity level for inductive and capacitive formulations. The conformity is defined on one hand for the magnetic flux density and the electric field, and on the other hand for the magnetic field and the electric flux density. For that purpose, a sequence of mixed FE spaces, i.e., with nodal, edge and face elements, is used for interpolating the considered fields and potentials. The global quantities involved in the circuit relations, i.e., the voltages, currents and charges, are given convenient discrete forms, allowing their natural coupling with the fields [7] and simplifying their evaluation. Examples of local and global solutions are given for a test problem.

Magnetic and Electric Coupled Models

A bounded domain Ω, of boundary $\partial\Omega = \Gamma$, of the three-dimensional Euclidean space is considered, in which the Maxwell equations are to be solved. The eddy current conducting part of Ω is denoted Ω_c and the non-conducting one Ω_c^C, with $\Omega = \Omega_c \cup \Omega_c^C$. Massive inductors belong to Ω_c.

The equations and relations governing the considered problem in Ω are

$$\operatorname{curl} \mathbf{h} = \mathbf{j} + \partial_t \mathbf{d} \, , \; \operatorname{curl} \mathbf{e} = -\partial_t \mathbf{b}, \tag{1a-b}$$

$$\operatorname{div} \mathbf{b} = 0 \, , \; \operatorname{div} \mathbf{d} = \rho, \tag{2a-b}$$

$$\mathbf{b} = \mu\, \mathbf{h}\,, \quad \mathbf{j} = \sigma\, \mathbf{e}\,, \quad \mathbf{d} = \varepsilon\, \mathbf{e}\,, \tag{3a-b-c}$$

where **h** is the magnetic field, **b** is the magnetic flux density, **e** is the electric field, **d** is the electric flux density, **j** is the electric current density, including source and eddy currents, r is the charge density, ? is the magnetic permeability, σ is the electric conductivity and ε is the electric permittivity. The current conservation equation is obtained from (1a), i.e.,

$$\operatorname{div}(\mathbf{j} + \partial_t \mathbf{d}) = 0\,. \tag{4}$$

An assumption consists in neglecting the displacement current $\partial_t \mathbf{d}$ in (1a), giving

$$\operatorname{curl} \mathbf{h} = \mathbf{j}\,, \tag{5}$$

but not in (4). This way, the equations to be solved can be split into two sub-models: the magnetodynamic model, defined by (5), (1b) in Ω_c, (2a), (3a) and (3b) in Ω_c, coupled with an electric model, defined by (1b), (2b) and (3c).

The magnetodynamic model only determines **e** in Ω_c and the electric model aims at its calculation in $\Omega_c{}^C$. The voltage applied to a conductor is first defined for the magnetodynamic model and is a source for the electric model, together with the magnetic solution to be used in the right hand side of (1b). Both inductive and capacitive currents have to be added to give the total current through (4), which couples the two sub-models.

Magnetodynamic and electric coupled models are intended to be weakly formulated for the FE method. Two kinds of conformity levels can be considered and are developed hereafter. It will be particularly shown that the conformity chosen for one model automatically fixes the one for the other coupled model.

Magnetic Flux Density and Electric Field Conform Formulations

Weak formulations, local fields and strong voltages

The common way to assure the conformity for **b** and **e**, expressing the conservation of both magnetic fluxes and electromotive forces, is to express the electric field e in Wc via a magnetic vector potential a together with the gradient of an electric scalar potential v, and **b** in Ω as the curl of this vector potential **a**, i.e.,

$$\mathbf{e} = -\partial_t \mathbf{a} - \operatorname{grad} \mathrm{v} \quad \text{in } \Omega_c\,, \tag{6}$$

$$\mathbf{b} = \operatorname{curl} \mathbf{a} \quad \text{in } \Omega\,. \tag{7}$$

This way, (1b) in Ω_c and (2a) are strongly satisfied. With these potentials, the **a**-v magnetodynamic formulation is obtained from the weak form of the Ampere equation (5), i.e. [5],

$$(\mu^{-1} \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega} + (\sigma\, \partial_t \mathbf{a}, \mathbf{a}')_{\Omega_c} + (\sigma \operatorname{grad} \mathrm{v}, \mathbf{a}')_{\Omega_c} + \langle \mathbf{n} \times \mathbf{h}_s, \mathbf{a}' \rangle_{\Gamma_h} = 0\,, \quad \forall\, \mathbf{a}' \in F_a(\Omega)\,, \tag{8}$$

where $\mathbf{n} \times \mathbf{h}_s$ is a constraint associated with certain boundaries Γ_h of domain Ω and $F_a(\Omega)$ is the curl-conform function space defined on Ω and containing the basis functions for **a** as well as for the test function **a**'; $(\cdot\,,\cdot)_\Omega$ and $<\cdot\,,\cdot>_\Gamma$ respectively denote a volume integral in Ω and a surface integral on Γ of the product of their vector field arguments. To assure the uniqueness of **a**, the function space $F_a(\Omega)$ has to be constrained with a gauge condition in $\Omega_c{}^C$, defined here at the discrete level by the tree co-tree technique [5], only keeping edge degrees of freedom for the co-tree edges.

For the electric model, the conformity for **e** is kept by extending the definition (6) to $\Omega_c{}^C$, using the magnetodynamic solutions **a** in Ω and v in Ω_c as sources. The term $-\partial_t \mathbf{a}$ can then be considered as a source electric field, the distribution of which being known in $\Omega_c{}^C$ by (8), while $-\operatorname{grad} \mathrm{v}$ is the unknown reaction field that should allow the electric flux density conservation (4). The known distribution of the scalar potential v in Ω_c and so on $\partial\Omega_c$ will serve as a boundary condition for $\Omega_c{}^C$.

Note that the field **e** in both Ω_c and Ω_c^C is non-conservative. This means that the potential v, giving the reaction field to the non-physical gauged source $-\partial_t \mathbf{a}$, does not have any physical meaning.

A way to express the electric model can be done through the weak form of (4), i.e.,

$$(\sigma \partial_t \mathbf{a}, \operatorname{grad} v')_{\Omega_c} + (\sigma \operatorname{grad} v, \operatorname{grad} v')_{\Omega_c} + (\varepsilon \partial_t^2 \mathbf{a}, \operatorname{grad} v')_{\Omega}$$
$$+ (\varepsilon \partial_t \operatorname{grad} v, \operatorname{grad} v')_{\Omega} = < \mathbf{n} \cdot (\mathbf{j} + \partial_t \mathbf{d}), v' >_{\Gamma_j}, \quad \forall v' \in F_v(\Omega), \tag{9}$$

where Γ_j is the part of the boundary of Ω_c which is crossed by a current, **n** is the unit normal vector exterior to Ω and $F_v(\Omega)$ is the grad-conform function space defined on Ω and containing the basis functions for v and the associated test function v'. The test function grad v' in (9) is a particular form of the test function **a**' in (8); this property is kept at the discrete level if nodal and edge FEs are used for v and **a** respectively [5]. Consequently, the sum of the first two terms of (9) is equal to zero for all the basis functions of v' with a zero trace on Γ_j. The remaining of the equation is actually the time derivative of the weak form of (2b). Equation (9) is also the weak form of (2b), derivated in time, added to (8) with $\mathbf{a}' = -\operatorname{grad} v'$ and $\Gamma_j \subset \Gamma_h$, the latter also being the weak form of $\operatorname{div} \mathbf{j} = 0$ in Ω_c.

Weak inductive and capacitive currents

From (9), the circuit relation gathering both weakly defined inductive and capacitive currents, the sum of which is the total current I_i flowing through the portion $\Gamma_{j,i}$ of a massive conductor i, can then be expressed as

$$I_i = (\sigma \partial_t \mathbf{a}, \operatorname{grad} v_{s,i})_{\Omega_c} + V_i (\sigma \operatorname{grad} v_{s,i}, \operatorname{grad} v_{s,i})_{\Omega_c} + (\varepsilon \partial_t^2 \mathbf{a}, \operatorname{grad} v_{s,i})_{\Omega} + (\varepsilon \partial_t \operatorname{grad} v, \operatorname{grad} v_{s,i})_{\Omega}, \tag{10}$$

where $v_{s,i}$, used as test function, is a global basis function for the voltage V_i, having the general property to be equal to 1 on one current gate and zero on all the others; it thus gives the surface integral term in (9) the value of the total current I_i.

At the discrete level, the function $v_{s,i}$ can be reduced in Ω_c to the sum of the nodal basis functions of all the nodes located on $\Gamma_{j,i}$ with a support limited to a transition layer (composed of the elements having nodes on $\Gamma_{j,i}$). Such a global function restricted to a reduced part of Ω_c was first defined in [5] in magnetodynamics and is extended here in Ω_c^C.

The first three integrals in (10) are known from the magnetodynamic model while the fourth one needs the solution of the electric model. The reduced support of $v_{s,i}$ allows the integrals in (10) to be limited to a few elements: these of its transition layer in Ω_c and these sharing faces with this layer in Ω_c^C.

Magnetic Field and Electric Flux Density Conform Formulations

Weak formulations, local fields and strong currents

The conformity for **h**, expressing the conservation of its circulation or of the currents, can be assured through the definition of a magnetic scalar potential ϕ in Ω_c^C, with $\mathbf{h} = -\operatorname{grad} \phi$. This potential is multivalued when Ω_c^C is multiply connected, in which case surface cuts must be defined to make this domain simply connected [6]. In Ω_c, the conformity for **h** is assured by a suitable definition of its function space, through edge FEs at the discrete level for strongly expressing (5).

The magnetodynamic (**h**-conform) **h**-ϕ formulation is obtained from the weak form of the Faraday equation, i.e. [6],

$$(\partial_t (\mu \mathbf{h}), \mathbf{h}')_{\Omega} + (\sigma^{-1} \operatorname{curl} \mathbf{h}, \operatorname{curl} \mathbf{h}')_{\Omega_c} + < \mathbf{n} \times \mathbf{e}_s, \mathbf{h}' >_{\Gamma_e} = 0, \quad \forall \mathbf{h}' \in F_{h\phi}(\Omega), \tag{11}$$

where $\mathbf{n}\times\mathbf{e}_s$ is a constraint associated with certain boundaries Γ_e of domain Ω and $F_{h\phi}(\Omega)$ is the function space defined on Ω and containing the basis functions for $\mathbf{h}$ and ϕ as well as for the test function $\mathbf{h}'$. For each global basis functions $\mathbf{c}_i$ associated with an inductive current (having a unit circulation around the associated inductor), the surface integral term in (11), with Γ_e the boundary of a voltage source, defines the voltage V_i acting on this inductor [6], which gives

$$(\partial_t(\mu\mathbf{h}),\mathbf{c}_i)_\Omega+(\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{c}_i)_{\Omega_c}=-<\mathbf{n}\times\mathbf{e}_s,\mathbf{c}_i>_{\Gamma_e}=-V_i\,. \quad (12)$$

For the coupled electric model, the conformity for $\mathbf{d}$ can be assured through the definition of an electric vector potential $\mathbf{u}$, with

$$\mathbf{d}=\operatorname{curl}\mathbf{u}\,, \quad (13)$$

thus satisfying (2b) with $\rho=0$. In case ρ differs from zero, a source electric flux density $\mathbf{d}_s$ would have to be defined [8], [9], with $\mathbf{d}=\mathbf{d}_s-\operatorname{curl}\mathbf{u}$ and $\operatorname{div}\mathbf{d}_s=\rho$. The electric model to be posed in $\Omega_c{}^C$ can only be given information regarding the tangential electric field on $\partial\Omega_c$ via a natural boundary condition. An essential condition on the tangential component of $\mathbf{u}$ would indeed contradict the joint consideration of (4) and (5): with

$$\sigma^{-1}\,\mathbf{n}\times\mathbf{h}=\varepsilon^{-1}\,\mathbf{n}\times\mathbf{u} \quad \text{on } \partial\Omega_c\,, \quad (14)$$

the circulation of $\mathbf{u}$ around a current gate could only be related to the inductive current without any possible additional displacement current.

The electric model is governed by the weak form of (1b) in $\Omega_c{}^C$, i.e.,

$$(\varepsilon^{-1}\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{u}')_{\Omega_c{}^C}+(\partial_t(\mu\mathbf{h}),\mathbf{u}')_{\Omega_c{}^C}+<\mathbf{n}\times\mathbf{e}_s,\mathbf{u}'>_{\Gamma_e+\partial\Omega_c}=0\,,\quad \forall\mathbf{u}'\in F_u(\Omega_c{}^C)\,, \quad (15)$$

where function space $F_u(\Omega_c{}^C)$ contains $\mathbf{u}$ and its associated test function $\mathbf{u}'$ and has to be constrained with a gauge condition. At the discrete level, $\mathbf{u}$ is discretised with edge FEs and is associated a gauge condition by the tree co-tree technique.

The natural boundary condition on the electric field on $\partial\Omega_c$ appears in the surface integral term of (15). Its expression for each test function $\mathbf{u}'$ can be directly given by (11) written only for Ω_c, i.e.,

$$(\partial_t(\mu\mathbf{h}),\mathbf{u}')_{\Omega_c}+(\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{u}')_{\Omega_c}+<\mathbf{n}\times\mathbf{e}_s,\mathbf{u}'>_{\partial\Omega_c}=0\,. \quad (16)$$

Adding (15) and (16), with normals $\mathbf{n}$ exterior to $\Omega_c{}^C$ and Ω_c respectively, thus of opposite signs, gives the resulting equation, i.e.,

$$(\varepsilon^{-1}\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{u}')_{\Omega_c{}^C}+(\partial_t(\mu\mathbf{h}),\mathbf{u}')_\Omega+<\mathbf{n}\times\mathbf{e}_s,\mathbf{u}'>_{\Gamma_e}+(\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{u}')_{\Omega_c}=0\,,\quad \forall\mathbf{u}'\in F_u(\Omega)\,, \quad (17)$$

illustrating well how the magnetic field acts as a source for determining $\mathbf{u}$ in $\Omega_c{}^C$ and $\partial\Omega_c$.

In addition, on the complementary part Γ_e in Γ of the charge or current gates Γ_j, with $\Gamma=\Gamma_e\cup\Gamma_j$, the electric vector potential $\mathbf{u}$ is subject to the essential boundary condition

$$\mathbf{n}\cdot\mathbf{d}\big|_{\Gamma_e}=\mathbf{n}\cdot\operatorname{curl}\mathbf{u}\big|_{\Gamma_e}=0\,, \quad (18)$$

which can be fulfilled through the definition of a surface scalar potential w associated with $\mathbf{u}$, i.e.,

$$\mathbf{n}\times\mathbf{u}\big|_{\Gamma_e}=-\mathbf{n}\times\operatorname{grad}w\big|_{\Gamma_e}\,. \quad (19)$$

This potential w is multivalued on Γ_e and must be given discontinuities, strongly linked with the currents through the gates, across cut lines on Γ_e. Such cut lines can be easily defined as the traces on Γ_e of the surface cuts defining the magnetic scalar potential discontinuities.

Weak voltages

Each global basis function describing a discontinuity of w defines a voltage V_i when applied as test function to the surface integral term in (17). This leads to the circuit relation between the voltage V_i, weakly defined, and the capacitive current, strongly expressed in **u**, i.e.,

$$(\varepsilon^{-1}\,\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{c}_i)_{\Omega_c{}^C} + (\partial_t(\mu\,\mathbf{h}), \mathbf{c}_i)_{\Omega} + (\sigma^{-1}\,\mathrm{curl}\,\mathbf{h}, \mathrm{curl}\,\mathbf{c}_i)_{\Omega_c} = -V_i\,. \qquad (20)$$

Relations (12) and (20) written jointly for an inductor give its impedance. Here again, thanks to the reduced support of each $\mathbf{c}_i$, the integrals to be evaluated are limited to a few elements: these of a transition layer on one side of each cut in $\Omega_c{}^C$ and these sharing faces with this layer in Ω_c.

Application

A micro-coil is considered as a test problem (Fig. 1). The coil is made of copper and its width and thickness are 5 μm. The gap between successive wires is 5 μm. It serves as an illustrative example showing the global basis functions used in each formulation to be associated with voltages or currents (Fig. 2), as presented in the theoretical part.

The complementarity of the dual solutions has been verified for the local fields, in particular the magnetic flux density (Fig. 3), the electric field (Figs. 4 and 5), as well as for the global quantities, i.e., the currents, voltages, resistances (Fig. 6), inductances (Fig. 7) and capacitances (Fig. 8). These quantities have been checked to tend toward limit values with the refinement of the mesh, for a wide frequency range.

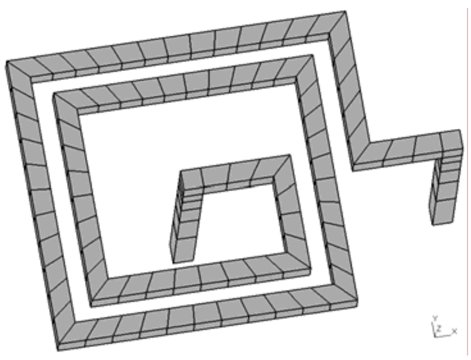

Fig. 1. Geometry of the micro-coil (coarse FE mesh) with its outputs.

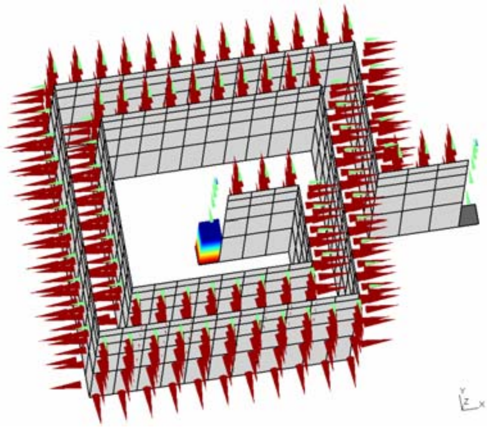

Fig. 2. Source scalar potential $v_{s,i}$ for the **a**-v formulation (support reduced to a tube starting at the left electrode) and cut surface (preventing any loop around the coil tube) for the **h** and **u** formulations with the associated global basis function $\mathbf{c}_i$ distribution (having a unit circulation around any coil tube).

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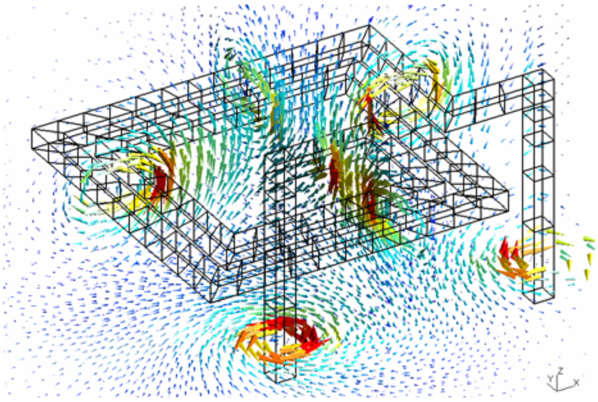

Fig. 3. Magnetic flux density distribution in cut planes.

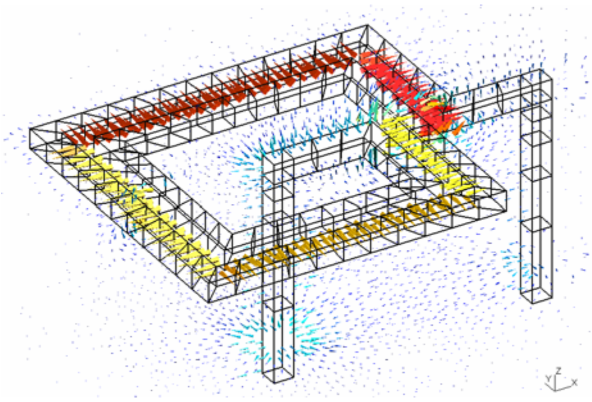

Fig. 4. Electric field distribution in cut planes.

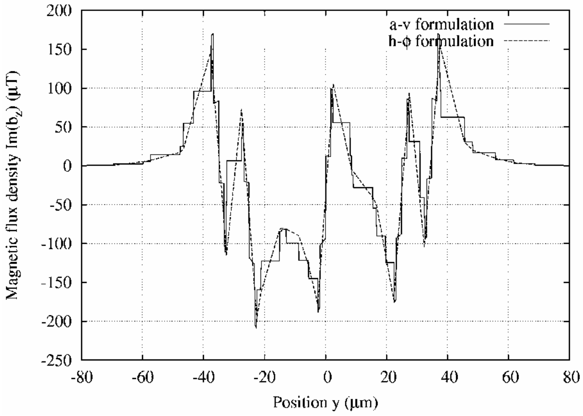

Fig. 5. Magnetic flux density in the coil plane ($Im(b_z)$) along a line crossing 5 wires).

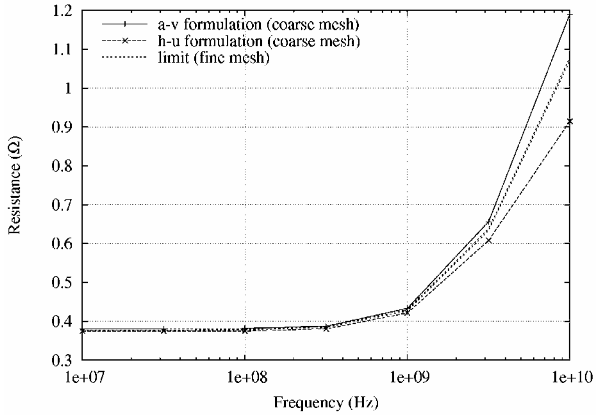

Fig. 6. Resistance versus frequency.

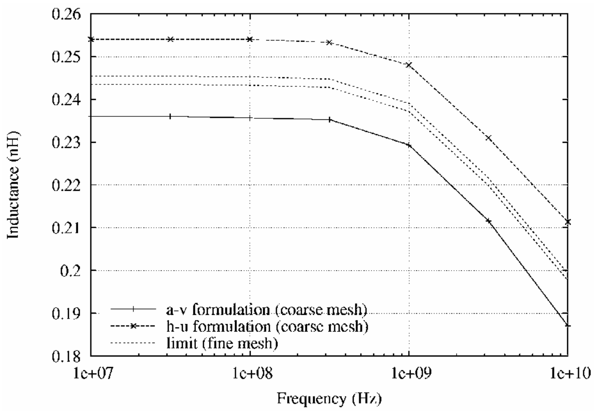

Fig. 7. Inductance versus frequency.

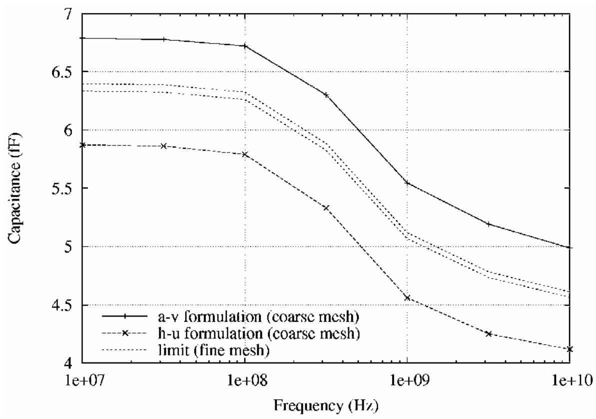

Fig. 8. Capacitance versus frequency.

Conclusions

This contribution points out the interest of satisfying conformity properties for magnetic and electric problems coupling inductive and capacitive effects in massive inductors. It develops the related mathematical and discretisation tools for any wished conformity level. In particular, it justifies the use of the non-common **d**-conform electric formulation [8], [9], when associated with the **h**-conform magnetodynamic formulation. In all cases, the basis functions associated with the global quantities involved in the FE circuit relations benefit from a significant support reduction, which facilitates their evaluation and gives direct physical interpretations of their expressions. The complementarity of dual solutions for a given problem gives the possibility to estimate the discretisation error.

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